QCD Inequalities, Large N_C and $\pi\pi$ Scattering Lengths

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Abstract

In this short note we show that (I) in a QCD-like theory with four (rather than two) degenerate flavors $ud\,u'd'$, the $\pi\pi'$ scattering length is positive (attractive); and (II) in QCD with only two (u,d) degenerate flavors the I=2 (say, $\pi^+\pi^+$ hadronic) scattering length is, in the large N_C limit, repulsive. $\pi(\pi')$ are the lowest physical states coupling to $J^p = \bar{u}(x)\gamma_5 d(x)$ and $J^{p'} = \bar{u}'(x)\gamma_5 d'(x)$, respectively.

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To probe the threshold $\pi\pi$ (or $(\pi'\pi')$) scattering consider the Euclidean two-point correlators:

$$Q(x) = < 0|J^p(x)J^p(x)(J^p(0)J^p(0))^+|0 > \tag{1}$$

$$Q'(x) = < 0|J^p(x)J^{p'}(x)(J^p(0)J^{p'}(0))^+|0 > \tag{2}$$

where $J^p = \bar{u}(x)\gamma_5 d(x)$ and $J^{p'} = \bar{u}'(x)\gamma_5 d'(x)$, respectively.

The correlators have spectral decompositions

$$Q(x) = \int_{\mu_0}^{+\infty} \sigma_Q(\mu^2)\exp(-\mu x)d\mu^2 \tag{3}$$

$$Q'(x) = \int_{\mu_0}^{+\infty} \sigma_Q'(\mu^2)\exp(-\mu x)d\mu^2 \tag{4}$$

with a common threshold $\mu_0 = 2m_\pi$ (or $\mu_0 = m_\pi + m_{\pi'}$). The asymptotic behavior of $Q(x)\,(Q'(x))$ in the limit $|x| \to \infty$ (or Euclidear time separating the points $(0 = (\vec{0},0)$ and $x = (\vec{0},t)$, $t \to \infty$) is controlled by the spectral density σ_Q or $\sigma_{Q'}$ at $\mu = 2m_\pi$ (or $\pi\pi'$) scattering at threshold can be described using nonrelativistic methods. The Levinson's theorem implies that the density of states in any convenient large radius R of the $\pi\pi\,(\pi\pi')$ system deviates from that of a free-noninteracting $\pi\pi\,(\pi\pi')$ pair by

$$\frac{dn_{\pi\pi}}{d\mu^2} - \frac{dn_{\pi\pi}^{(0)}}{d\mu^2} = \frac{d\delta(k)^{(L=0)}}{dk} \approx a^{(L=0)} \tag{5}$$

with $k = 1/2\sqrt{\mu^2 - 4m_\pi^2}$ the CMS momentum, and $\delta^{(0)}$ and $a^{(0)}$ are the S-wave phase, shift and scattering length of the $\pi\pi$ system.

Analogously for $\pi\pi'$,

$$\frac{dn_{\pi\pi'}}{d\mu^2} - \frac{dn_{\pi\pi'}^{(0)}}{d\mu^2} = \frac{d\delta(k)^{(L=0)}}{dk} \approx a'^{(L=0)}. \tag{6}$$

Because hadronic forces have finite range only the $L = 0$ wave needs to be considered. By definition,

$$\frac{dn_{\pi\pi}}{d\mu^2} \propto \sigma_Q(\mu^2); \quad \frac{dn_{\pi\pi'}}{d\mu^2} \propto \sigma_Q'(\mu^2). \tag{7}$$

To prove (I), we will show that

$$Q'(x) \geq < 0|J^p(x)J^{p^+}(0)|0 > < 0|J^{p'}(x)J^{p'^+}(0)|0 > . \tag{8}$$

That is, the joint propagation of the four quarks—which asymptotically becomes the joint π and π' propagation from 0 to x—is enhanced relative to the product of the separate $J^p J^p$ and $J^{p'} J^{p'}$ correlators. The latter is in fact the independent π and π' propagations between 0 and x.

To prove Eq. (8), we use the path integral representation of the various correlators. Since all the four-quark flavors created at 0 and annihilated at x are distinct, there is a unique "contraction" for $Q'(x)$ or Eq. (2) yielding:

$$Q'(x) = \int d\mu(A) \text{ tr } \gamma_5 S_{u_A}(x,0)\gamma_5 S_{d_A}(0,x) \cdot \text{tr } \gamma_5 S_{u'_A}(x,0)\gamma_5 S_{d'_A}(0,x) \qquad (9)$$

$$= \int d\mu(A)\text{tr}(S_A^+(0,x)S_A(0,x)) \cdot \text{tr } (S_A^+(0,x)S_A(0,x))$$

where

$$d\mu(A) = D(A_\mu(x)) \cdot e^{-SYM\{A_\mu(x)\}}\Pi_{u,d,u'd'} Det(\slashed{D}_A^{q_i} + m_i)/Z. \qquad (10)$$

is the positive [1,2] measure in the path integral. $\slashed{D}_A = \gamma_\nu(\delta_\nu + gA_\mu^a\lambda_a)$ is the covariant Dirac operator in the $A_\mu^a(x)$ background, and $S_{i_A}(x) = < 0|\frac{1}{\slashed{D}_A+m_i^0}|x >$ is the propagator of the Fermion (i.e., quark of flavor i and bare mass m_i^0) in the $A_\mu^a(x)$ background. Finally the partition function Z in the denominator of Eq. (10) normalizes the path integral measure to $\int d\mu(A) = 1$.

In deriving the second line of Eq. (9), we have used the "γ_5 conjugation" property of the Euclidean propagator,

$$\gamma_5 \, S_A^i(0,x)\gamma_5 = [S_A^i(x,0)]^\dagger \qquad (11)$$

with the $\dagger$ referring here to Hermitian conjugation in color-spinor space of the $S_A^i(x,0)_{\alpha\alpha',aa'}$ 12×12 matrix with spinor $1 \le \alpha, \alpha' \le 4$ and color $1 \le a, a' \le N_C$ indices, which for simplicity we generally omit. The latter γ_5 conjugation also implies the positivity of the determinant of the Dirac operator $(\slashed{D}_A + m^0)$, which is the key to the claimed positivity of the measure $d\mu(A)$. The proofs of all these appear in the original paper of Weingarten [1] and of Vafa and Witten [2] and in a recent comprehensive review [3]. Finally, we utilized the fact that since $m_u^0 = m_d^0 = m_{u'}^0 = m_{d'}^0$, all four propagators are actually the same.

$$S_A^u(x,0) = S_A^d(x,0) = S_A^{u'}(x,0) = S_A^{d'}(x,0) \equiv S_A(x,0). \qquad (12)$$

The inequality

$$Q'(x) \ge P(x)P'(x) \qquad (13)$$

with

$$P(x) = < 0|J^p(x)(J^p(0))^+|0 > ; \;\; P'(x) = < 0|J^{p'}(x)(J^{p'}(0))^+|0 > \qquad (14)$$

can be readily derived [3]. To this end we compare $Q'(x)$—as given by the path integral (second line of Eq. (9))—and the product of path integrals for $P(x), P'(x)$:

$$P(x) = \int d\mu(A)\text{tr}(S_A^+(0,x)S_A(0,x)) \qquad (15)$$

and

$$P'(x) = \int d\mu(A)\text{tr}(S_A'^+(0,x)S_A'(0,x)) \qquad (16)$$

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(which in the $m_u^{(0)} = m_{u'}^{(0)}; m_d^{(0)} = m_{d'}^{(0)}$ case of interest are equal as $S_A = S_A'$).

Let us denote

$$\text{tr } S_A^+(0, x) S_A(0, x) \equiv \pi_A(x) \geq 0. \tag{17}$$

The desired inequality (13) then is (as most QCD correlator inequalities are) just the Schwartz inequality;

$$\int d\mu(A) \pi_A(x) \pi_A(x) \geq |\int d\mu(A) \pi_A(x)|^2. \tag{18}$$

One dramatic way in which Eq. (18) could be implemented is if a four-quark *bound* (scalar?) doubly-charged state $\bar{u}\gamma_5 d \ \bar{u}'\gamma_5 d$ existed below the $\pi^+\pi'^+ (= 2m_\pi)$ threshold [4]. We are trying to mimic real QCD where we know that exotic $q\bar{q}q\bar{q}$ mesons—should they exist at all—are much heavier than $2m_\pi \approx 270$ MeV. Hence we will *assume* that there are no such bound states below threshold.

Using then Eqs. (5)-(7) above, our inequality (18) implies an attractive (positive) S-wave $\pi\pi'$ scattering length; namely, assertion I above. One can also directly argue by going to the nonrelativistic limit and viewing the path integral expression as an euclidean diffusive evolution, that the probability of returning to the origin $\vec{r} = 0$ after some long time T is enhanced *iff* the interaction between the $\pi\pi'$ is attractive. [5] A related, more ambitious approach uses four-point inequalities [6,7].

One might argue that the introduction of π' (and the corresponding extra two flavors, $u'd'$, degenerate with u and d) is unphysical and renders this result meaningless. Such an argument could be even more forcefully made against Weingarten's proof [1] of $m_N \geq m_\pi$ where <u>six</u> degenerate yet distinct flavors were introduced.

We believe that this is not the case. In particular, the augmentation of the flavor sector can be useful for picking up specific flavor contraction patterns in the real QCD with two (or three) light (almost) degenerate flavors. Thus, let us consider $\pi\pi$ scattering in QCD (see Fig. 1).
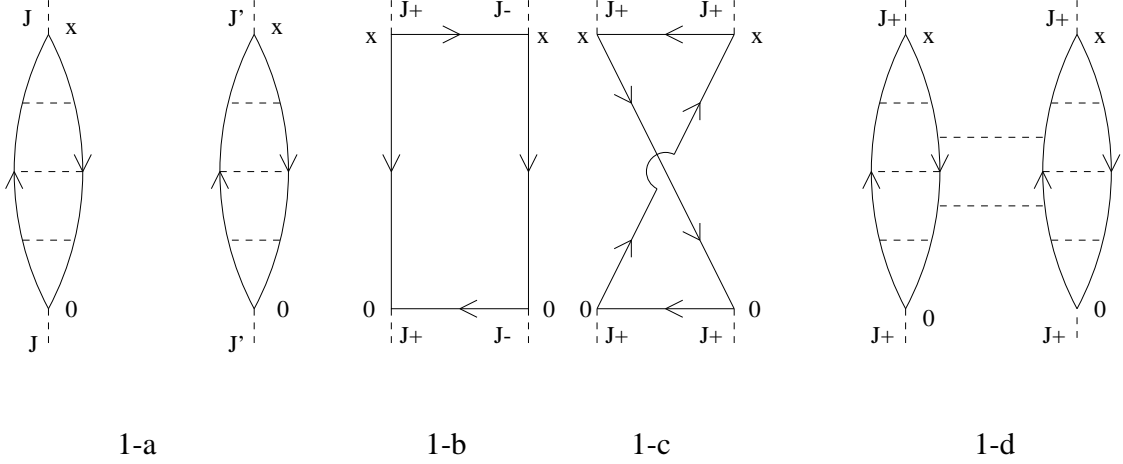
FIGURES

| 1-a | 1-b | 1-c | 1-d |

FIG. 1. Fig. 1a: Independent propagation of the $u\bar{d}$ and $u'd'$ blobs corresponding to the $\pi\pi'$ case. Fig. 1b: The anihilation contraction diagram for the case of $\pi^+\pi^-$. Fig. 1c: The exchange contraction diagram for the case of $\pi^+\pi^+$. Fig. 1d: The multi-gluon exchange diagram.

We can have separate, independent propagation of two pions, represented in Fig. 1(a), where we have gluons exchanged only between $u\bar{d}$ (or $\bar{u}d$) in the same initial and final pions. For $\pi^+\pi^-$ scattering we have the flavor connected contraction depicted in Fig. 1(b), corresponding to $q\bar{q}$ annihilation. Likewise for $\pi^+\pi^+$ scattering, we have instead the quark exchange diagram, Fig. 1(c). Finally, for all pion pairs—and for the $\pi\pi'$ case—we can have the $\bar{q}q$ bubbles interaction also via gluon exchanges (at least two gluons are required because of the color neutrality), as indicated in Fig. 1(d).

A key point is that for $\pi\pi'$ scattering we have only the free separate propagation (Fig. 1(a)) and gluon exchanges (Fig. 1(d)): the π and π' have no common quarks and/or antiquarks to allow for $\bar{q}q'$ annihilation and/or $\dot{q}q'$ exchange.

The same arguments could be made in any vectorial theory and, in particular, in QED. This then conforms to the attractive perturbative two-photon exchange, $1/r^6$ van der Waals (VDW) potential and its "retarded" Casimir Polder version at large distances. For two polarizable atoms A, B the latter is

$$V_{AB}^{CP} = -\frac{\hbar c}{(4\pi)^3 r^7}[23(\alpha_A^E \alpha_B^E + \alpha_A^M \alpha_B^M) - 7(\alpha_A^E \alpha_B^M + \alpha_B^E \alpha_A^M)] \tag{19}$$

The VDW interaction is, by second-order perturbation, always attractive between two stationary systems in their ground state. The CP interaction is manifestly positive if A and B are dynamically the same, i.e., $\alpha_A^E = \alpha_B^E$ and $\alpha_A^M = \alpha_B^M$.

Amusingly, the case for which we have been able to generalize this result satisfies both requirements. First, we know (and it can also be proven via QCD inequalities) that the pseudoscalar π (and π') are indeed the lightest mesons. Second, since $m_u^0 = m_d^0(= m_{u'}^0 = m_{d'}^0)$ and the gauge interaction are flavor independent, the scattering "A and B", i.e., π and π (or π and π') are dynamically the same. We note that V_{CP} of Eq. (19) remains attractive so long as the objects A and B have similar ratios of electrical and magnetical polarizabilities. This is expected to be the case for the "chromo" polarizabilities of different hadrons, suggesting

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that $(J/\psi - A, Z)(D_s - A, Z)(D_s, B_u)$, etc., bound states of heavy/extended hadrons sharing no common quark flavors will form.

We next turn to our second main goal; namely, point II above. With only u and d degenerate quark flavors we have for $\pi^+\pi^-$ ($\pi^+\pi^+$) scattering, or, for the correlators:

$$< 0|J^{p_s+}(x)J^{p_s-}(x)(J^{p_s+}(0)J^{p_s-}(0)^+|0 >= Q_{\pi^+\pi^-} \tag{20}$$

$$< 0|J^{p_s+}(x)J^{p_s+}(x)(J^{p_s+}(0)J^{p_s+}(0)^+|0 >= Q_{\pi^+\pi} , \tag{21}$$

the contribution of the additional contractions. These are

$$Q_{\pi^+\pi^-}(x)|_{Annihilation} = \int d\nu(A)\mathrm{tr}\, S_A^{+d}(0,x)S_A^u(x,x)S_A^d(x,0)S_A^{u+}(0,0) \tag{22}$$

and

$$Q_{\pi^+\pi^+}(x)|_{Exchange} = -\int d\nu(A)\mathrm{tr}\{S_A^+(0,x)S(0,x)S_A^+(0,x)S_A^d(x,0)S_A(0,x)\}, \tag{23}$$

respectively. The crucial minus sign in Eq. (23) stems from the need to permute two $\psi_u(x)$ (or two $\psi_u(0)$) noncommuting operators so as to arrive from the $\int d\nu(A)\mathrm{tr}\{S_A^+(0,x)S_A^+(0,x)\}^2$ contraction pattern of Fig. 1(a) + 1(d) to Eq. (23) and the contraction 1(c).

In the large N_C limit the contributions (22) and (23) to the correlators dominate. It is tempting to associate these—in the chiral, threshold limit—with the classical [8] results, $a^{I=0} \approx 0, 2\, m_\pi^{-1}$ and $a^{I=2} = -\frac{2}{7}\, a^{I=0}$. Higher-order corrections in the chiral expansions preserve the negataive (repulsive) $a^{I=2}$—in agreement with experimental measurements which unfortunately are rather poor for $a^{I=2}$. This is particularly gratifying in view that we have an opposite sign contribution (due to the multi-gluon, Fig. 1(d)) which does not, however, reverse the sign. Let us conclude with two comments.

(i) The expressions (22) and (23) for $Q_{\pi^+\pi^-}$ and $Q_{\pi^+\pi^+}$ seem drastically different. However, in the threshold chiral limit they may both converge to similar objects. The point is that as (x) (or t) tends to ∞, the relevant propagating hadronic system is totally dominated by the threshold pion states. In the spirit of "dual-string-QCD" approach, which may not be completely inappropriate [9,10] we may then represent the soft pions as two nearby $u\bar{d}$ lines so that their mass proportional to the string bit length between them approaches zero and they are indeed point-like as appropriate for the chiral Lagrangian approach. Naively, we would then expect the diagrams corresponding to Figs. 1(b) and 1(c) to be essentially equal up to a factor (-1). This would be the case if in the VDM spirit we can describe the process merely via a ρ exchange. In this case, $a^{I=2} = -\frac{1}{2}\, a^{I=1}$. To get a better ratio we need some $I = 0$ scalar t channel exchange which enhances $a_{\pi^+\pi^-}$ and suppresses $a_{\pi^+\pi^+}$. The magnitude of this term is fixed by requiring that it will have the same magnitude also in the crosses, s, channel, thereby obtaining the desired -2/7 ratio of the $I = 2$ and $I = 0$ scattering length.

(ii) We can estimate the contribution of the multi-gluon exchanges to threshold $\pi\pi$ physics by the positive shift of $a^{I=0}$ and $a^{I=2}$ as compared with the original chiral Lagrangian estimate. It is tempting to identify the term,

$$4\pi a^2_{gluon\,exchange} \tag{24}$$

with the asymptotic $\pi^+\pi^+$ cross section (controlled also by multi-gluon exchange) extrapolates all the way to threshold. We will pursue this in a future publication.

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